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The existence of an asymptotic thermally developed region for laminar forced convection in a circular duct

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Abstract—With reference to the stationary and laminar forced convection in a circular duct for an incompressible Newtonian fluid having a fully developed velocity profile, previous studies on the existence of an asymptotic thermally developed region are improved. A sufficient condition, much broader than those previously found in the literature for the existence of an asymptotic thermally developed region, is determined. It is proved that this condition is also necessary if one requires that the asymptotic value of the Nusselt number is nonzero. A numerical computation of the temperature field in the thermal entrance region for two different wall heat flux distributions which fulfil the sufficient condition for the existence of an asymptotic thermally developed region and which yield the same asymptotic value of the Nusselt number is performed. Copyright © 1996 Elsevier Science Ltd.

INTRODUCTION

As is well known [1], several experiments on forced convection in circular ducts have pointed out that the Nusselt number tends to become invariant along the flow direction under certain boundary conditions, such as when either the wall temperature or the wall heat flux is uniform. This circumstance has raised the problem of determining the necessary and sufficient condition for the existence of an asymptotic thermally developed region. In the literature [2–5], it is commonly accepted that this necessary and sufficient condition is that the wall heat flux varies, at least asymptotically, with an exponential law.

In a recent paper [6], a different sufficient condition for the existence of an asymptotic thermally developed region is found. Namely, it is shown that if the axial distribution of wall heat flux $q_w(x)$ is such that $(1/q_w)$ dq_w/dx tends to zero for $x \to +\infty$, then both the local Nusselt number and the dimensionless temperature profile $\vartheta = (T_w - T)/(T_w - T_b)$ become asymptotically invariant along the duct. Moreover, the asymptotic value of the Nusselt number equals 48/11, as in the case of uniform wall heat flux. Indeed, this constraint on the asymptotic behaviour of $q_w(x)$ is satisfied by many axial distributions $q_w(x)$, which for large values of x are neither uniform nor exponentially varying, such as power-law wall heat fluxes [6].

The aim of this paper is to determine a sufficient condition for the existence of an asymptotic thermally developed region, which includes as particular cases both the condition obtained in ref. [6] and the usually accepted condition that the wall heat flux behaves exponentially for large values of x. The sufficient condition determined here is also necessary if one requires that the asymptotic value of the Nusselt number is nonzero. Reference is made to stationary and laminar forced convection in a circular duct for an incompressible Newtonian fluid with constant properties, with a fully developed velocity profile, negligible axial conduction and negligible viscous dissipation.

The paper is organized as follows. First, a constraint on the wall heat flux $q_w(x)$ is found, which represents a necessary condition for the existence of an asymptotic radial distribution of dimensionless temperature with a non-vanishing asymptotic value of the Nusselt number. Then, the boundary value problem which describes laminar forced convection in a circular duct for an incompressible Newtonian fluid with constant properties, with a fully developed velocity profile, negligible axial conduction and negligible viscous dissipation is analysed. A sufficient condition for the forced convection problem to allow an asymptotic thermally developed region is found. This condition is much broader than that obtained in ref. [6], and broader than the condition that the wall heat flux varies exponentially for large values of x. The asymptotic values of the Nusselt number implied by this condition are evaluated by a series method. Finally, the theoretical results are illustrated by a finite difference determination of the thermal entrance regions for two axial distributions of wall heat flux which fulfil the sufficient condition for the existence of an asymptotic thermally developed region, and which yield the same asymptotic value of the Nusselt number.

NOMENCLATURE

A_n	dimensionless coefficients employed in Appendix B
A	constant introduced in equation (26) [K]
c_n	dimensionless coefficients defined in Appendix B
f	solution of equation (22)
$\mathscr{F}$	function employed in Appendix A [m^3 K^2 s^{-1}]
k	thermal conductivity [W m^{-1} K^{-1}]
L	length of the tube [m]
L_{th}	thermal entry length [m]
L_{th}^*	dimensionless thermal entry length
N_r	number of grid intervals in direction r
N_x	number of grid intervals in direction x
Nu	Nusselt number, $2r_0 q_w/[k(T_w - T_b)]$
Pe	Peclet number, $2\bar{u}r_0/\alpha$
q_w, q_{w1}, q_{w2}	wall heat fluxes [W m^{-2}]
r	radial coordinate [m]
r_j	radial coordinate at grid position (j, N_x) [m]
Δr	length of each grid interval in direction r [m]
r_0	radius of the tube [m]
s	dimensionless radius, $s = r/r_0$
T	temperature [K]
T_1, T_2	temperature fields [K]
T_{j,N_x}	temperature at grid position (j, N_x) [K]
T_0, $T_0^{(1)}$, $T_0^{(2)}$	inlet temperature distributions [K]
$\tilde{T}$	temperature field, $\tilde{T} = T_1 - T_2$ [K]
$\mathscr{T}$, $\mathscr{T}_0$	solutions of equation (19)
u	velocity component in the axial direction [m s^{-1}]
$\bar{u}$	mean value of u [m s^{-1}]
x	axial coordinate [m]
x'	integration variable [m]
Δx	length of each grid interval in direction x [m].

Greek symbols

α	thermal diffusivity [m^2 s^{-1}]
β	constant defined by equation (5) [m^{-1}]
$\tilde{\beta}$	dimensionless parameter, $\tilde{\beta} = Pe\, r_0 \beta$
γ	constant employed in equations (35) and (36) [m^{-1}]
ϑ	dimensionless temperature $\vartheta = (T_w - T)/(T_w - T_b)$
ϑ_∞	dimensionless function defined by equation (4).

Subscripts

b	bulk quantity
w	quantity evaluated at the wall.

A NECESSARY CONDITION FOR AN ASYMPTOTIC THERMALLY DEVELOPED REGION

In this section, a constraint on q_w which represents a necessary condition for the existence of an asymptotic thermally developed region with a non-vanishing Nusselt number is determined.

Hereafter, reference will be made to the stationary and laminar forced convection in a circular duct for an incompressible Newtonian fluid with constant properties, with a fully developed velocity profile, negligible axial conduction and negligible viscous dissipation. Therefore, the energy equation will be written as

$$\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{r u(r)}{\alpha}\frac{\partial T}{\partial x} \tag{1}$$

where $0 \leqslant r \leqslant r_0$, $x \geqslant 0$, and the axial component u of the velocity field is given by the Hagen–Pouiseuille expression,

$$u(r) = 2\bar{u}\left(1 - \frac{r^2}{r_0^2}\right). \tag{2}$$

The forced convection problem described above allows an asymptotic thermally developed region if, for every r,

$$\operatorname*{Lim}_{x\to+\infty}\frac{\partial}{\partial x}\left(\frac{T_w - T}{T_w - T_b}\right) = \operatorname*{Lim}_{x\to+\infty}\frac{\partial\vartheta}{\partial x} = 0 \tag{3}$$

and if there exists a function $\vartheta_\infty(r)$ such that, for every r,

$$\operatorname*{Lim}_{x\to+\infty}\frac{T_w - T}{T_w - T_b} = \operatorname*{Lim}_{x\to+\infty}\vartheta = \vartheta_\infty(r). \tag{4}$$

It is easily proved that if ϑ becomes asymptotically invariant then also the Nusselt number $Nu = 2r_0 q_w/[k(T_w - T_b)]$ becomes asymptotically invariant, and that the asymptotic value of Nu is equal to $-2r_0\, \mathrm{d}\vartheta_\infty/\mathrm{d}r|_{r=r_0}$ [7].

Let us now prove that if the forced convection problem allows a thermally developed region and if the asymptotic value of the Nusselt number is non-vanishing, then there exists a real constant β such that

$$\operatorname*{Lim}_{x\to+\infty}\frac{1}{q_w}\frac{\mathrm{d}q_w}{\mathrm{d}x} = \beta. \tag{5}$$

On account of the definition of Nusselt number, q_w can be expressed as

$$q_w = \frac{k\,Nu}{2r_0}(T_w - T_b). \tag{6}$$

By employing equation (6), ϑ can be rewritten as

$$\vartheta = \frac{k}{2r_0} Nu \frac{T_w - T}{q_w}. \tag{7}$$

The derivation with respect to x of both sides of equation (7) yields

$$\frac{\partial T}{\partial x} = \frac{dT_w}{dx} - \frac{2r_0}{k}\left[\frac{\vartheta}{Nu}\frac{dq_w}{dx} - \frac{q_w\vartheta}{Nu^2}\frac{dNu}{dx} + \frac{q_w}{Nu}\frac{\partial\vartheta}{\partial x}\right]. \tag{8}$$

The bulk temperature and the bulk dimensionless temperature are given by

$$T_b(x) = \frac{2}{\bar{u}r_0^2}\int_0^{r_0} T(r,x)u(r)r\,dr \tag{9}$$

$$\vartheta_b(x) = \frac{2}{\bar{u}r_0^2}\int_0^{r_0} \vartheta(r,x)u(r)r\,dr. \tag{10}$$

Since $\vartheta = (T_w - T)/(T_w - T_b)$, equations (9) and (10) yield $\vartheta_b(x) = 1$, for every x. Therefore, on account of equations (8)–(10), one obtains

$$\frac{dT_w}{dx} = \frac{dT_b}{dx} + \frac{2r_0}{k}\left[\frac{1}{Nu}\frac{dq_w}{dx} - \frac{q_w}{Nu^2}\frac{dNu}{dx}\right]. \tag{11}$$

The substitution of equation (11) into equation (8) yields

$$\frac{\partial T}{\partial x} = \frac{dT_b}{dx} - \frac{2r_0}{k}\left[\frac{\vartheta - 1}{Nu}\frac{dq_w}{dx} - \frac{q_w(\vartheta-1)}{Nu^2}\frac{dNu}{dx} + \frac{q_w}{Nu}\frac{\partial\vartheta}{\partial x}\right]. \tag{12}$$

By employing the energy balance equation [1]

$$\frac{dT_b}{dx} = \frac{2\alpha}{k\bar{u}r_0}q_w \tag{13}$$

equation (12) can be rewritten as

$$\frac{\partial T}{\partial x} = \frac{2\alpha}{k\bar{u}r_0}q_w - \frac{2r_0}{k}\left[\frac{\vartheta - 1}{Nu}\frac{dq_w}{dx} - \frac{q_w(\vartheta-1)}{Nu^2}\frac{dNu}{dx} + \frac{q_w}{Nu}\frac{\partial\vartheta}{\partial x}\right]. \tag{14}$$

The substitution of equations (7) and (14) into equation (1) yields

$$\frac{\partial}{\partial r}\left(r\frac{\partial\vartheta}{\partial r}\right) = -\frac{ru(r)}{\alpha}\left[\frac{\alpha Nu}{\bar{u}r_0^2} - \frac{\vartheta-1}{q_w}\frac{dq_w}{dx} + \frac{\vartheta-1}{Nu}\frac{dNu}{dx} - \frac{\partial\vartheta}{\partial x}\right]. \tag{15}$$

Since the forced convection problem allows a thermally developed region, equation (4) ensures that, when x tends to infinity, the left hand side of equation (15) tends to a finite limit. On the other hand, on account of equation (3), when x tends to infinity, $\partial\vartheta/\partial x$ tends to zero. Moreover, since the asymptotic Nusselt number is non-vanishing, $(1/Nu)\ dNu/dx$ tends to zero and the derivative of $\vartheta_\infty(r)$ at $r = r_0$ is non-vanishing. Therefore, when x tends to infinity, $\vartheta - 1$ cannot be identically vanishing in the interval $0 \leqslant r \leqslant r_0$. As a consequence, the right hand side of equation (15) tends to a finite limit when x tends to infinity, only if there exists a real constant β such that equation (5) holds.

Note that *not only* the wall heat fluxes $q_w(x)$ expressed by an exponential function fulfil equation (5). Indeed, with proper values of β, equation (5) is satisfied by all polynomial functions, rational functions (i.e. ratios between polynomials), hyperbolic functions and the logarithmic function. On the contrary, equation (5) does not hold for wall heat fluxes $q_w(x)$ expressed by trigonometric functions and functions such as x^x, x^{-x}, e^{x^2}, e^{-x^2}.

A SUFFICIENT CONDITION FOR AN ASYMPTOTIC THERMALLY DEVELOPED REGION

In this section, a sufficient condition for the forced convection problem to allow an asymptotic thermally developed region is determined.

If both the inlet temperature distribution and the axial distribution of wall heat flux are prescribed, the energy equation yields the boundary value problem

$$\begin{cases} \dfrac{\partial}{\partial r}\left(r\dfrac{\partial T}{\partial r}\right) = \dfrac{ru(r)}{\alpha}\dfrac{\partial T}{\partial x} \\ k\left.\dfrac{\partial T}{\partial r}\right|_{r=r_0} = q_w(x) \\ T(r,0) = T_0(r) \end{cases} \tag{16}$$

In Appendix A, it is proved that equation (16) has a unique solution. The proof is similar to that of the uniqueness theorem for the initial value problem of heat conduction presented in Carslaw and Jaeger [8]. It can be easily proved that the uniqueness of the solution of the boundary value problem (16) holds also if the boundary condition at $r = r_0$ is of the first kind, i.e. if the prescribed quantity at $r = r_0$ is the temperature.

Let $T_0^{(1)}(r)$ and $T_0^{(2)}(r)$ be two different inlet temperature distributions. Let us denote by $T^{(1)}(x,r)$ the solution of equation (16) with $T_0(r) = T_0^{(1)}(r)$, and by $T^{(2)}(x,r)$ the solution of equation (16) with $T_0(r) = T_0^{(2)}(r)$. It has been proved in ref. [6] that the difference $T^{(1)}(x,r) - T^{(2)}(x,r)$ tends to a constant when x tends to infinity. Therefore, neither the asymptotic behaviour of Nu, nor that of the dimensionless temperature ϑ can be influenced by the inlet temperature distribution. In fact, both Nu and ϑ are defined through temperature differences. As a consequence of these results, the analysis of the asymptotic behaviour of the dimensionless temperature field can be performed by considering the reduced boundary value problem

$$\begin{cases} \dfrac{\partial}{\partial r}\left(r\dfrac{\partial T}{\partial r}\right) = \dfrac{ru(r)}{\alpha}\dfrac{\partial T}{\partial x} \\ k\left.\dfrac{\partial T}{\partial r}\right|_{r=r_0} = q_w(x) \end{cases}. \quad (17)$$

Equation (17) has infinite solutions: one for every prescribed inlet temperature distribution. Obviously, any solution can be expressed in the form

$$T(r,x) = \frac{r_0}{k\,Pe}\left[\mathscr{T}(r,x)q_w(x) + \frac{4}{r_0}\int_0^x q_w(x')\,dx'\right], \quad (18)$$

where $Pe = 2\bar{u}r_0/\alpha$ is the Peclet number. On account of equations (17) and (18), function $\mathscr{T}(r,x)$ must be such that

$$\begin{cases} \dfrac{\partial}{\partial r}\left(r\dfrac{\partial \mathscr{T}}{\partial r}\right) = \dfrac{ru(r)}{\alpha}\left[\dfrac{1}{q_w}\dfrac{dq_w}{dx}\mathscr{T} + \dfrac{\partial \mathscr{T}}{\partial x} + \dfrac{4}{r_0}\right] \\ \left.\dfrac{\partial \mathscr{T}}{\partial r}\right|_{r=r_0} = \dfrac{Pe}{r_0} \end{cases}. \quad (19)$$

If the wall heat flux $q_w(x)$ fulfils equation (5), let us assume that there exists a solution of equation (19), $\mathscr{T}_0(r,x)$, such that, for every r,

$$\lim_{x\to+\infty} \frac{\partial \mathscr{T}_0(r,x)}{\partial x} = 0 \quad (20)$$

$$\lim_{x\to+\infty} \mathscr{T}_0(r,x) = f(r) \quad (21)$$

where $f(r)$ is an analytic function of r. This assumption is legitimate. In fact, by employing equations (5), (20) and (21), in the limit $x\to+\infty$, equation (19) yields

$$\begin{cases} \dfrac{d}{dr}\left(r\dfrac{df}{dr}\right) = \dfrac{ru(r)}{\alpha}\left(\beta f + \dfrac{4}{r_0}\right) \\ \left.\dfrac{df}{dr}\right|_{r=r_0} = \dfrac{Pe}{r_0} \end{cases}. \quad (22)$$

For the case $\beta \neq 0$, in Appendix B it is proved that there exists a solution of equation (22) which is regular in $r = 0$ and is given by

$$f(r) = Pe\frac{\sum_{n=0}^{\infty} c_n (r/r_0)^{2n}}{2\sum_{n=0}^{\infty} nc_n} - \frac{4}{\beta r_0}. \quad (23)$$

The coefficients c_n are defined by the recursion formula

$$c_0 = 1$$
$$c_1 = \frac{\tilde{\beta}}{4}$$
$$c_n = \frac{\tilde{\beta}}{(2n)^2}(c_{n-1} - c_{n-2}), \quad n \geq 2 \quad (24)$$

where $\tilde{\beta} = Pe\, r_0\beta$. In Appendix B, it is proved that the power series which appears on the right hand side of equation (23) has an infinite radius of convergence.

On the other hand, in the case $\beta = 0$, it is easily proved that a solution of equation (22) is given by

$$f(r) = \frac{Pe}{r_0^2}\left(r^2 - \frac{r^4}{4r_0^2} - \frac{7}{24}r_0^2\right). \quad (25)$$

Therefore, for high values of x, any solution $T(r,x)$ of the reduced boundary value problem (17) can be expressed as

$$T(r,x) = A + \frac{r_0}{k\,Pe}\left[f(r)q_w(x) + \frac{4}{r_0}\int_0^x q_w(x')\,dx'\right] \quad (26)$$

where A is a real constant which depends on the inlet temperature distribution. By employing equations (22) and (25), it can be easily checked that the bulk value of the function $f(r)$, given by

$$f_b = \frac{2}{\bar{u}r_0^2}\int_0^{r_0} f(r)u(r)r\,dr \quad (27)$$

is zero for every value of β. Therefore, the limit for $x\to+\infty$ of the dimensionless temperature ϑ is given by

$$\lim_{x\to+\infty} \vartheta(r,x) = \lim_{x\to+\infty} \frac{T_w(x) - T(r,x)}{T_w(x) - T_b(x)}$$
$$= \frac{f(r_0) - f(r)}{f(r_0)} = \vartheta_\infty(r). \quad (28)$$

On account of equations (23) and (25)–(28), $\vartheta_\infty(r)$ can be expressed as

$$\vartheta_\infty(r) = \frac{\sum_{n=1}^{\infty} c_n\left[1 - \left(\dfrac{r}{r_0}\right)^{2n}\right]}{\sum_{n=0}^{\infty} c_n\left(1 - \dfrac{8n}{\tilde{\beta}}\right)} \quad (29)$$

if $\beta \neq 0$, and as

$$\vartheta_\infty(r) = \frac{24}{11}\left(\frac{3}{4} - \frac{r^2}{r_0^2} + \frac{r^4}{4r_0^4}\right) \quad (30)$$

if $\beta = 0$. Equation (29) yields an analytic function of r for every real value of $\tilde{\beta}$ which does not correspond to a zero of $f(r_0)$. Moreover, it can be easily proved that $\vartheta_\infty(r)$ is a continuous function of the parameter $\tilde{\beta}$, for $\tilde{\beta} = 0$. In the following section, it will be shown that in the range $-100 \leq \tilde{\beta} \leq 100$ there is a value of

$\tilde{\beta}$, namely $\tilde{\beta} = -40.5549$, which corresponds to a zero of $f(r_0)$.

On account of equations (25) and (28), the limit for $x \to +\infty$ of $\partial\vartheta/\partial x$ is given by

$$\lim_{x\to+\infty} \frac{\partial\vartheta(r,x)}{\partial x} = \lim_{x\to+\infty} \left[\frac{\dfrac{dT_w(x)}{dx} - \dfrac{\partial T(r,x)}{\partial x}}{T_w(x)-T_b(x)} - \vartheta(r,x) \frac{\dfrac{dT_w(x)}{dx} - \dfrac{dT_b(x)}{dx}}{T_w(x)-T_b(x)} \right]$$

$$= \lim_{x\to+\infty} \left\{ \frac{\dfrac{dq_w(x)}{dx}[f(r_0)-f(r)]}{q_w(x)f(r_0)} - \frac{\vartheta(r,x)}{q_w(x)} \frac{dq_w(x)}{dx} \right\}$$

$$= \beta\vartheta_\infty(r) - \beta\vartheta_\infty(r) = 0. \quad (31)$$

Equation (31) ensures that equation (3) is fulfilled.

Finally, on account of equations (24), (29) and (30), it can be easily checked that two boundary value problems with two wall heat fluxes $q_{w1}(x)$ and $q_{w2}(x)$ which satisfy equation (5) and have the same value of $\tilde{\beta}$ determine the same function $\vartheta_\infty(r)$. An interesting consequence of this result is that the asymptotic Nusselt number, which can be evaluated as

$$\lim_{x\to+\infty} Nu = -2r_0 \left. \frac{d\vartheta_\infty}{dr} \right|_{r=r_0} \quad (32)$$

does not depend on the detailed functional structure of $q_w(x)$, but depends only on $\tilde{\beta} = Pe\, r_0 \beta$, where β is given by equation (5). In particular, equations (29) and (32) yield

$$\lim_{x\to+\infty} Nu = \frac{4\sum_{n=1}^{\infty} nc_n}{\sum_{n=0}^{\infty} c_n\left(1-\dfrac{8n}{\tilde{\beta}}\right)}, \quad (33)$$

while equations (30) and (32) yield

$$\lim_{x\to+\infty} Nu = \frac{48}{11}. \quad (34)$$

On account of equation (34), any axial distribution $q_w(x)$ which fulfils equation (5) with $\beta = 0$ yields an asymptotic value of Nu equal to 48/11, as it has been proved in ref. [6]. A series expression of the asymptotic value of the Nusselt number equivalent to that given by equation (33) was employed by Roetzel [11] in the analysis of laminar forced convection in a circular duct with exponentially varying wall heat flux. In this paper, equation (33) assumes a broader field of application.

FULLY DEVELOPED TEMPERATURE FIELD

In this section, the asymptotic behaviour of the temperature field and of the Nusselt number for wall heat fluxes which fulfil equation (5) is analysed.

In the preceding section, it has been proved that two different wall heat fluxes which fulfil equation (5) and have the same value of $\tilde{\beta}$ have also the same asymptotic value of the Nusselt number and the same asymptotic distribution of dimensionless temperature. Equation (5), for any prescribed value of β, is satisfied by exponentially varying wall heat fluxes $q_w(x) = q_0 \exp(\beta x)$. Therefore, for any axial distribution of wall heat flux $q_w(x)$ which fulfils equation (5), it is possible to find an exponentially varying wall heat flux with the same asymptotic value of the Nusselt number and the same asymptotic distribution of dimensionless temperature. Indeed, the asymptotic values of Nu for exponentially varying wall heat fluxes have been determined by various authors, either in the absence of axial heat conduction in the fluid [2, 3, 11, 12] or in the presence of this effect [13]. In Table 1, asymptotic values of Nu evaluated by equations (33) and (34) in the range $-100 \leqslant \tilde{\beta} \leqslant 100$ are reported and compared with those obtained in ref. [3] with reference to exponentially varying wall heat flux. The values reported in ref. [3] agree with those obtained by employing equations (33) and (34). The evaluation performed in ref. [3] is based on a numerical determination of eigenvalues and eigenfunctions and on the computation of a series whose convergence is very slow. On the other hand, the series which appears in equation (33) has a very fast convergence: 1000 terms are sufficient to evaluate Nu in the range

Table 1. Asymptotic values of Nu for various values of $\tilde{\beta}$ compared with those obtained by Shah and London [3] for exponential wall heat flux

$\tilde{\beta}$	Nu	Nu (Shah and London [3])
−100	−12.0000	—
−90	−2.5748	—
−80	1.2135	—
−70	3.7532	—
−60	6.3291	—
−50	11.3470	—
−40	−130.7177	—
−30	−2.2989	−2.29
−20	1.6834	1.68
−10	3.3423	3.34
0	4.3636	4.364
10	5.1067	5.11
20	5.6974	5.71
30	6.1925	6.21
40	6.6222	6.64
50	7.0040	7.02
60	7.3493	—
70	7.6655	—
80	7.9582	—
90	8.2313	—
100	8.4877	—

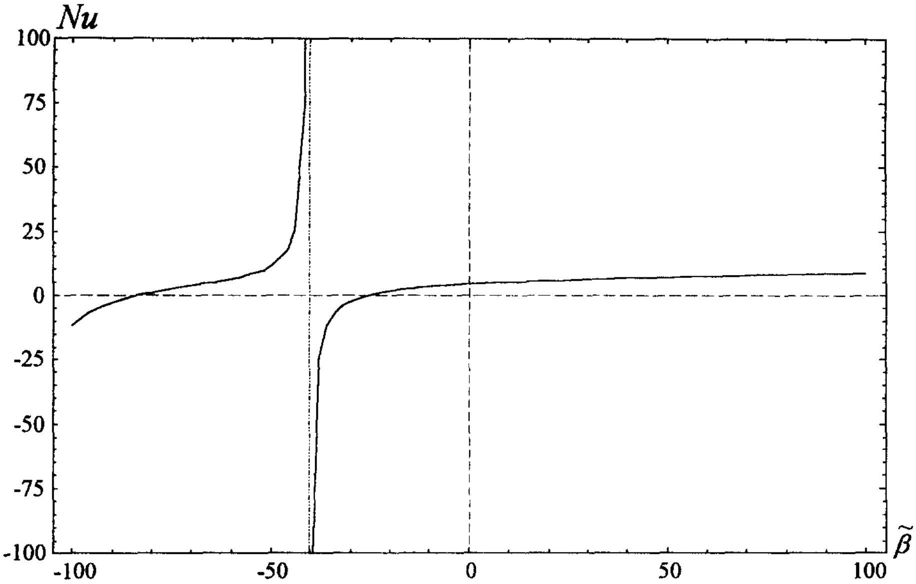

Fig. 1. Plot of the asymptotic value of Nu as a function of $\tilde{\beta}$.

$-100 \leqslant \tilde{\beta} \leqslant 100$ with 10 decimal digits accuracy. A plot of the asymptotic value of Nu as a function of $\tilde{\beta}$ is reported in Fig. 1. A singularity occurs for $\tilde{\beta} = -40.5549$. For this value of $\tilde{\beta}$, both the Nusselt number and the dimensionless temperature distribution fail to develop. This and other singularities of the asymptotic Nusselt number which may occur are not determined by any pathological behaviour of the asymptotic temperature field. Indeed, Nu becomes asymptotically singular whenever $T_w - T_b$ goes to zero for $x \to +\infty$. Since f_b is identically vanishing, equation (26) implies that $T_w - T_b$ goes to zero for $x \to +\infty$ if $\tilde{\beta}$ is such that $f(r_0) = 0$. In Fig. 2, a plot of $f(r)/Pe$ for $\tilde{\beta} = -40.5549$ is reported: this value of $\tilde{\beta}$ yields $f(r_0) = 0$. Figure 1 shows also that in the range $-100 \leqslant \tilde{\beta} \leqslant 100$ the asymptotic value of Nu becomes zero for two values of $\tilde{\beta}$: these values are $\tilde{\beta} = -83.8618$ and $\tilde{\beta} = -25.6796$. The latter value agrees with that predicted in ref. [14], i.e. $\tilde{\beta} = -25.68$, obtained by an interpolation of the asymptotic values of Nu determined in ref. [3] for exponentially varying wall heat flux.

THE THERMAL ENTRANCE REGION

In this section, an illustration of the results obtained in the preceding sections is performed by the numerical evaluation of the temperature field in the thermal entrance region for two different axial distributions of

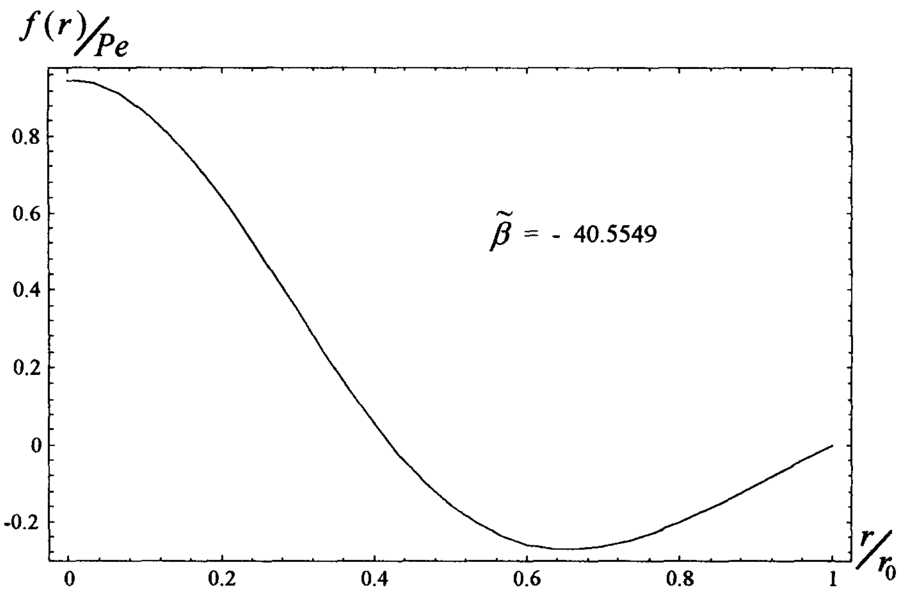

Fig. 2. Plot of $f(r)/Pe$ for $\tilde{\beta} = -40.5549$.

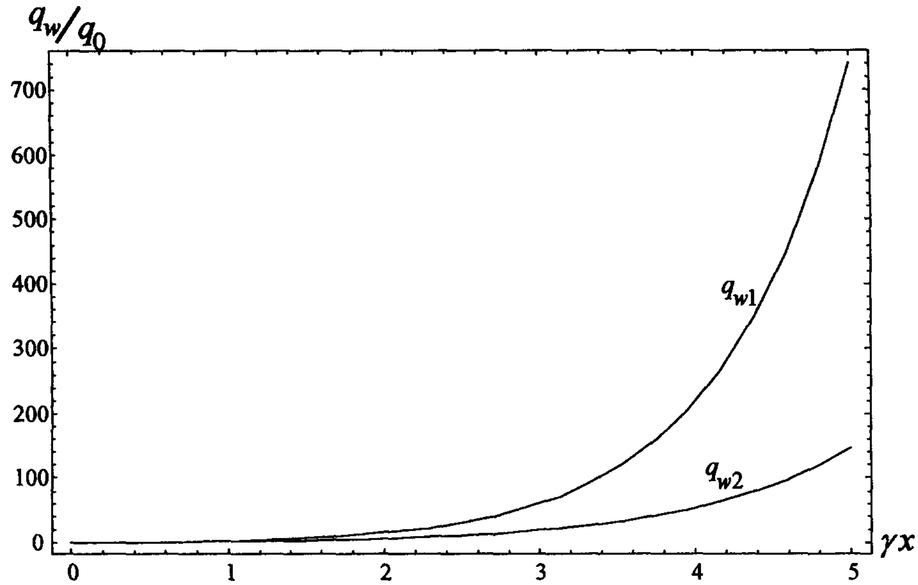

Fig. 3. Plots of q_{w1} and of q_{w2} as functions of γx.

wall heat flux, which satisfy equation (5) and which have the same value of $\tilde{\beta}$.

Let us consider the axial distributions of wall heat flux defined as

$$q_{w1}(x) = q_0 \gamma x \exp(\gamma x), \quad (35)$$

$$q_{w2}(x) = q_0[\exp(\gamma x) - 1]. \quad (36)$$

The axial variations of wall heat flux determined by equations (35) and (36) are represented in Fig. 3. The plots of $q_{w1}(x)$ and $q_{w2}(x)$ reported in Fig. 3 show that these axial distributions differ especially for high values of x. However, as it can be easily checked, both $q_{w1}(x)$ and $q_{w2}(x)$ fulfil equation (5) with $\beta = \gamma$; therefore, they should yield the same asymptotic value of the Nusselt number.

Under the assumption that the inlet temperature distribution is uniform with value T_0, the thermal entrance region has been determined with a finite difference method, both when the wall heat flux is given by equation (35) and when it is given by equation (36). The values of the parameters have been chosen in order to obtain $\tilde{\beta} = 10$. It is easily verified that, for each wall heat flux, the local value of Nu as a function of the dimensionless axial coordinate $x/(2r_0 Pe)$ is uniquely determined by the value of $\tilde{\beta}$.

The finite difference method employs a two-dimensional grid which is uniform both with respect to r and to x. The region $0 \leqslant x \leqslant L$ is subdivided in N_x intervals with length $\Delta x = L/N_x$, while the region $0 \leqslant r \leqslant r_0$ is subdivided in N_r intervals with length $\Delta r = r_0/N_r$. The computation has been performed with $N_r = 100$ and $N_x = 600$. The details of the finite difference method are described in ref. [6].

The bulk temperature can be evaluated exactly by employing equation (13). When the wall heat flux is given by equation (35), an integration of equation (13) yields

$$T_b(x) = T_0 + \frac{2\alpha q_0}{k\bar{u} r_0 \gamma}[(\gamma x - 1)\exp(\gamma x) + 1], \quad (37)$$

while, when the wall heat flux is given by equation (36), one obtains

$$T_b(x) = T_0 + \frac{2\alpha q_0}{k\bar{u} r_0 \gamma}[\exp(\gamma x) - \gamma x - 1]. \quad (38)$$

Equations (37) and (38) are employed to check if the global energy balance is satisfied by the numerical solution. In other words, the exact value of $T_b(L)$, evaluated either by equation (37) or by equation (38), is compared with the value of $T_b(L)$ obtained by a discrete sum approximation of the radial integral in equation (9), namely

$$T_b(L) \cong \frac{4\Delta r}{r_0^2} \sum_{j=1}^{N_r - 1} T_{j,N_x}\left(1 - \frac{r_j^2}{r_0^2}\right) r_j. \quad (39)$$

This comparison has shown that the relative errors in the quantity $T_b(L) - T_0$ due to the numerical approximation are less than 1.3%.

A comparison between the thermal entrance regions for the two wall heat fluxes given by equations (35) and (36) is presented in Fig. 4. This figure shows that the thermal entry length is higher when the wall heat flux is given by equation (35). The thermal entry length L_{th} is usually defined as the duct length required to achieve a value of local Nusselt number equal to 1.05 times its fully developed value [3]. The dimensionless thermal entry length is defined as $L_{th}^* = L_{th}/(2r_0 Pe)$. When the wall heat flux is given by

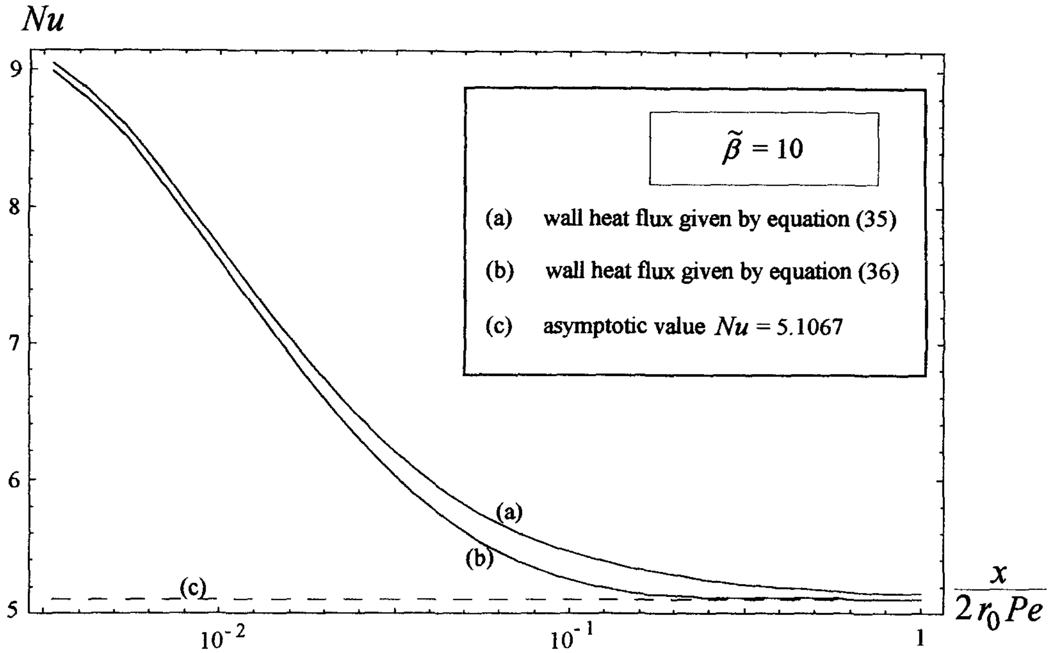

Fig. 4. Plots of the thermal entrance regions obtained with $q_w = q_{w1}$ and with $q_w = q_{w2}$.

equation (35), $L^*_{th} = 0.1376$, while, when the wall heat flux is given by equation (36), $L^*_{th} = 0.0755$.

Figure 4 shows that both wall heat fluxes yield the same asymptotic value of the Nusselt number. The exact value, reported in Table 1 for $\tilde{\beta} = 10$, is 5.1067.

CONCLUSIONS

The stationary and laminar forced convection in a circular duct for an incompressible Newtonian fluid with constant properties, with a fully developed velocity profile, negligible axial conduction and negligible viscous dissipation has been considered.

A constraint on the wall heat flux $q_w(x)$ which represents a necessary condition for the forced convection problem to allow an asymptotic thermally developed region with a non-vanishing Nusselt number has been determined. This constraint requires that $(1/q_w)\, dq_w/dx$ tends to a real constant β when $x \to +\infty$, and is fulfilled not only by exponential wall heat fluxes, but also by functions $q_w(x)$ which cannot be approximated by an exponential function, even for large values of x. Moreover, it has been proved that the above mentioned constraint determines also a sufficient condition for the forced convection problem to allow an asymptotic thermally developed region. More precisely, it has been proved that if the wall heat flux is such that $(1/q_w)\, dq_w/dx$ tends to a real constant β when $x \to +\infty$, and if the dimensionless parameter $\tilde{\beta}$ is not a zero of $f(r_0)$, both the Nusselt number and the dimensionless temperature profile tend to an asymptotically invariant and non-vanishing limit when $x \to +\infty$. The asymptotic values of the Nusselt number implied by this condition are uniquely determined by the dimensionless parameter $\tilde{\beta}$ and have been evaluated by a series method in the range $-100 \leqslant \tilde{\beta} \leqslant 100$. It has been shown that, in this range, the asymptotic value of the Nusselt number becomes singular when $\tilde{\beta} = -40.5549$. Finally, a finite difference computation of the temperature field in the thermal entrance region for two axial distributions of wall heat flux has been performed. These axial distributions fulfil the sufficient condition for the forced convection problem to allow an asymptotic thermally developed region with the same value of β. In agreement with the theoretical results obtained in this paper, the numerical computation confirms that, when $\tilde{\beta} = 10$, both wall heat fluxes yield the same asymptotic value of the Nusselt number, i.e. $Nu = 5.1067$.

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APPENDIX A

Let us prove that if $T_1(r,x)$ and $T_2(r,x)$ are two temperature fields which solve the boundary value problem (16), then $T_1(r,x) = T_2(r,x)$ for every r and for every x. In fact, it can be easily checked that the function $\tilde{T}(r,x) = T_1(r,x) - T_2(r,x)$ is a solution of the boundary value problem

$$\begin{cases} \dfrac{\partial}{\partial r}\left(r\dfrac{\partial \tilde{T}}{\partial r}\right) = \dfrac{ru(r)}{\alpha}\dfrac{\partial \tilde{T}}{\partial x} \\ \left.\dfrac{\partial \tilde{T}}{\partial r}\right|_{r=r_0} = 0 \\ \tilde{T}(r,0) = 0. \end{cases} \tag{A1}$$

Let us define a function $\mathscr{F}(x)$ as follows:

$$\mathscr{F}(x) = \int_0^{r_0} \tilde{T}(r,x)^2 u(r) r \, dr. \tag{A2}$$

Obviously, on account of equation (2), $\mathscr{F}(x) \geqslant 0$ for every $x \geqslant 0$. In particular, equations (A1) and (A2) ensure that $\mathscr{F}(0) = 0$. The derivative of $\mathscr{F}(x)$ is given by

$$\frac{d\mathscr{F}(x)}{dx} = 2\int_0^{r_0} \tilde{T}(r,x)\frac{\partial \tilde{T}(r,x)}{\partial x} u(r) r \, dr. \tag{A3}$$

By employing equation (A1), equation (A3) can be rewritten as

$$\frac{d\mathscr{F}(x)}{dx} = 2\alpha\int_0^{r_0} \tilde{T}(r,x)\frac{\partial}{\partial r}\left(r\frac{\partial \tilde{T}(r,x)}{\partial r}\right) dr. \tag{A4}$$

An integration by parts of the right hand side of equation (A4) yields

$$\frac{d\mathscr{F}(x)}{dx} = 2\alpha\left[r\tilde{T}(r,x)\frac{\partial \tilde{T}(r,x)}{\partial r}\right]_0^{r_0} - 2\alpha\int_0^{r_0} r\left(\frac{\partial \tilde{T}(r,x)}{\partial r}\right)^2 dr. \tag{A5}$$

By employing the condition of regularity of $\tilde{T}$ at $r = 0$, i.e. $\partial \tilde{T}/\partial r|_{r=0} = 0$, and the boundary condition on $\tilde{T}$ at $r = r_0$, equation (A5) yields

$$\frac{d\mathscr{F}(x)}{dx} = -2\alpha\int_0^{r_0} r\left(\frac{\partial \tilde{T}(r,x)}{\partial r}\right)^2 dr. \tag{A6}$$

Obviously, equation (A6) ensures that $d\mathscr{F}(x)/dx \leqslant 0$ for every $x \geqslant 0$, i.e. $\mathscr{F}(x)$ is a monotonically decreasing function of x. Since $\mathscr{F}(0) = 0$, one concludes that $\mathscr{F}(x) \leqslant 0$ for every $x \geqslant 0$. On the other hand, on account of equation (A2), $\mathscr{F}(x)$ is non-negative. Therefore, $\mathscr{F}(x) = 0$ for every $x \geqslant 0$. The integral on the right hand side of equation (A2) vanishes only if the integrand is zero, i.e. if $\tilde{T}(r,x) = 0$ for every r and for every x.

APPENDIX B

The boundary value problem given by equation (22) can be rewritten as

$$\begin{cases} \dfrac{d}{dr}\left(r\dfrac{df}{dr}\right) = \dfrac{2\bar{u}}{\alpha}\left(r - \dfrac{r^3}{r_0^2}\right)\left(\beta f + \dfrac{4}{r_0}\right) \\ \left.\dfrac{df}{dr}\right|_{r=r_0} = \dfrac{Pe}{r_0} \end{cases}. \tag{B1}$$

If the dimensionless radius $s = r/r_0$ is employed, the differential equation can be expressed as

$$s\frac{d^2 f}{ds^2} + \frac{df}{ds} + Pe\, r_0 \beta (s^3 - s)\left(f + \frac{4}{\beta r_0}\right) = 0. \tag{B2}$$

Equation (B2) has a regular singular point at $s = 0$ and can be solved through a power series by the method of Frobenius [9]. Let $f(r)$ be expressed by the power series

$$f(r) = -\frac{4}{\beta r_0} + s^m \sum_{n=0}^{\infty} A_n s^n. \tag{B3}$$

By substituting equation (B3) into equation (B2), one obtains on the left hand side of equation (B2) a power series whose coefficients must vanish. In particular, by equating to zero the coefficient of the lowest power of s, namely s^{m-1}, one obtains $m = 0$. By equating to zero the coefficients of the higher powers of s, one obtains a recursion relation which determines $A_0, A_1, \ldots, A_n, \ldots$, namely

$$\begin{aligned} A_{2n+1} &= 0, \quad n \geqslant 0 \\ A_2 &= \frac{\tilde{\beta}}{4} A_0 \\ A_4 &= \frac{\tilde{\beta}}{16}(A_2 - A_0) \\ A_{2n} &= \frac{\tilde{\beta}}{(2n)^2}(A_{2n-2} - A_{2n-4}), \quad n \geqslant 2. \end{aligned} \tag{B4}$$

Therefore, the power series contains only even powers of s and has an infinite radius of convergence as it can be easily inferred by employing the theorem A reported in Section 29 of Simmons [9]. Let us define the coefficients c_n as

$$c_n = \frac{A_{2n}}{A_0}, \quad n \geqslant 0. \tag{B5}$$

Since $m = 0$, by substituting equation (B5) in equation (B3) and by employing equation (B4), one obtains

$$f(r) = -\frac{4}{\beta r_0} + A_0 \sum_{n=0}^{\infty} c_n s^{2n}. \tag{B6}$$

The constant A_0 can be determined by imposing the boundary condition which appears in equation (B1), namely

$$\frac{A_0}{r_0} \sum_{n=0}^{\infty} 2nc_n = \frac{Pe}{r_0}. \tag{B7}$$

Since derivation does not modify the radius of convergence of a power series [10], the series which appears on the left hand side of equation (B7) is convergent. On account of equations (B6) and (B7), $f(r)$ is given by

$$f(r) = Pe \frac{\sum_{n=0}^{\infty} c_n (r/r_0)^{2n}}{2 \sum_{n=0}^{\infty} nc_n} - \frac{4}{\beta r_0}. \tag{B8}$$